Determination of Unstable Limit Cycles in Chaotic Systems by the Method of Unrestricted Harmonic Balance

Klaus Neymeyr and Friedrich Franz Seelig
Institut für Physikalische und Theoretische Chemie, Universität Tübingen, Tübingen

Z. Naturforsch. **46a**, 499–502 (1991); received March 8, 1991

The method of *unrestricted harmonic balance* (UHB) which is a generalization of the old method of harmonic balance and that was developed in preceding papers, is mathematically refined and applied to the evaluation of *unstable* limit cycles. The method is demonstrated for the case of the best investigated chaotic system, namely the Lorenz system. Some representative results are given.

1. Introduction

The phenomena found in some higher order systems of non-linear differential equations like strange attractors, chaotic oscillations, successive bifurcations etc. have attracted considerable interest in the last decade and contributed to an understanding of such fundamental topics like turbulence. The lowest dimension necessary to show these effects was found to be 3. One of the oldest and best investigated model systems is that of Lorenz [1]

$$\begin{aligned} \dot{x} &= \sigma(y-x), \\ \dot{y} &= x(r-z)-y, \\ \dot{z} &= xy-bz, \end{aligned} \tag{1}$$

that originally was established to understand fundamental phenomena of meteorology but evolved more and more as a purely mathematical object in its own right. An up to that time complete collection of known properties of that system was given by Sparrow [2]. Some results show the existence of a stable limit cycle around a stable critical point, so there must be an unstable limit cycle in between. But these and more complicated unstable limit cycles cannot be visualized by simulation, e.g. by a Runge-Kutta procedure. Sometimes it is possible to reverse the stability properties by reversing time. But this method works only if *all* non-zero Lyapunov exponents of the limit cycle have been positive before the time reversal, which is not the case here. So this trick cannot be applied. Here the method of *unrestricted harmonic balance* (UHB) fills the gap since its convergence is not determined by the stability of the system under investigation.

Reprint requests to Prof. Dr. F. F. Seelig, Institut für Physikalische und Theoretische Chemie, Universität Tübingen, Auf der Morgenstelle 8, W-7400 Tübingen.

2. The UHB Method as Applied to the Lorenz System

The old method of harmonic balance utilizes the fact that every periodic structure can be expanded into a Fourier series. If $\boldsymbol{x}(t)$ is periodic, i.e. for all t and proper T $\boldsymbol{x}(t+T)=\boldsymbol{x}(t)$, each component x_i of $\boldsymbol{x}$ can be represented by

$$x_i(t) = \sum_{j=0}^{\infty} A_{ij} \cos(j\omega t + \delta_{ij}) \tag{2}$$

with specific amplitudes A_{ij}, phase constants δ_{ij} and fundamental frequency

$$\omega = \frac{2\pi}{T}, \tag{3}$$

which is equivalent to

$$x_i(t) = \overline{x_i} + \sum_{j=1}^{\infty} (x_{cij} \cos(j\omega t) + x_{sij} \sin(j\omega t)). \tag{4}$$

In numerical treatments the Fourier series must be truncated at some finite but arbitrary harmonic $j=N$, yielding in both representations a $(2N+1)$-dimensional vector of characteristic constants for each x_i, since in (2) δ_{i0} is indefinite.

In most cases of former applications of the harmonic balance N was restricted to 1, which is a very poor approximation and can yield, in application to limit cycles of finite distance from the unstable critical point which it surrounds, fairly bad and sometimes even qualitatively wrong results.

In a previous series of papers [3–7] by one of us (F.F.S.) we removed this restriction for the case that the non-linearity consists of products of state variables x_i expanding thus the method to UHB. Since the highest harmonic $j=N$ can be shifted further in subsequent passes of the algorithm depending on the de-

0932-0784 / 91 / 0600-0499 $ 01.30/0. – Please order a reprint rather than making your own copy

sired accuracy, the truncation error is under complete control. In paper I [3] the principal ideas were developed: multiplication of two truncated Fourier series of order N yields another truncated Fourier series of order $2N$. But the method to be tractable needs a truncation at order N, too, introducing an additional error which is roughly of the same order as the former truncation error. In a computer program this special kind of vector product is introduced and executed in a subroutine that produces a new pseudo-linear variable $p(t)$ represented by a $(2N+1)$-dimensional vector in quite the same manner as the two $(2N+1)$-dimensional vectors of the factor variables $x(t)$ and $y(t)$, which can be symbolized by

$$p = x \odot y\,. \tag{4}$$

In [6] it was shown that by Taylor series expansion the method can be expanded even to non-linearities in the form of transcendental functions.

By this procedure the system of non-linear differential equations is transformed to a system of non-linear algebraic equations which can be solved iteratively by standard algorithms of numerical mathematics.

Since each variable is represented by a $(2N+1)$-vector, in the worst case the problem is of the order of $n(2N+1)$, if the system consists of n variables $x_i(t)$. But in nearly all cases this can be avoided if the system is treated in the following way:

1) rearrange to get as many as possible *linear* equations which allow to get simple expressions for one variable as a function of another,
2) search for *explicit non-linear* expressions,
3) the rest is *implicit non-linear* which yields a set of M non-linear equations of M unknowns that have to be solved iteratively.

The Lorenz system is a nice example to demonstrate this treatment. Since (1 a) is purely linear, y can be expressed by x in the form

$$y = x + \frac{\dot{x}}{\sigma}\,, \tag{5}$$

which by insertion of (4) for x and y and by collecting terms of $\sin(j\omega t)$, and $\cos(j\omega t)$ for each j separately and obeying that (5) should be true for every t yields

$$\bar{y} = \bar{x}$$

for $j = 1, \ldots, N$:

$$y_{cj} = x_{cj} + \frac{j\omega}{\sigma} x_{sj}\,, \quad y_{sj} = x_{sj} - \frac{j\omega}{\sigma} x_{cj}\,. \tag{6}$$

By forming the new pseudo-linear variable

$$p = x \odot y\,, \tag{7}$$

(1 c) is seen to be actually explicit non-linear,

$$b z + \dot{z} = p\,, \tag{8}$$

yielding in the same way

$$\bar{z} = \bar{p}/b$$

for $j = 1, \ldots, N$:

$$z_{cj} = \frac{b p_{cj} - j\omega p_{sj}}{b^2 + j^2 \omega^2}\,, \quad z_{sj} = \frac{b p_{sj} + j\omega p_{cj}}{b^2 + j^2 \omega^2}\,, \tag{9}$$

so that with the pseudo-linear

$$q = x \odot z\,, \tag{10}$$

(1 b) in the form

$$y + \dot{y} - r x + q = 0 \tag{11}$$

is left as a system of $(2N+1)$ actually implicit non-linear equations which yield iteratively from

$$\bar{y} - r\bar{x} + \bar{q} = 0$$

for $j = 1, \ldots N$:

$$y_{cj} + j\omega y_{sj} - r x_{cj} + q_{cj} = 0\,,$$
$$y_{sj} - j\omega y_{cj} - r x_{sj} + q_{sj} = 0\,, \tag{12}$$

the $(2N+1)$ unknowns $\bar{x}$, ω, $\{x_{c2},,,, x_{cN}\}$, $\{x_{sj}\}$ ($j = 1, \ldots, N$). Here x_{c1} can be removed because the time zero is arbitrary and can be fixed setting $x_{c1} = 0$. In [3–7] the algorithm of Powell [8] was used to solve a set of non-linear algebraic equations iteratively. In this work we used a more recent modified non-linear least squares algorithm of Oren-Luenberger class [9], whose FORTRAN code was given as ACM-Algorithm 573 NL2SOL by Dennis et al. [10].

3. Results

Depending on the chosen set of parameters σ, b, r, a number of stable and unstable limit cycles could be found. To get some order we chose for σ and b the same values as Lorenz did in his basic paper, namely $\sigma = 10$ and $b = 8/3$. Only r was varied, where the results of Sparrow were some guideline. A delicate question was the initial guess of the unknowns to be evaluated iteratively. Iterations were done in steps $N = 3$, 6, 12, 24, where in each loop the results of the former step served as an initial guess for the next, the values for the

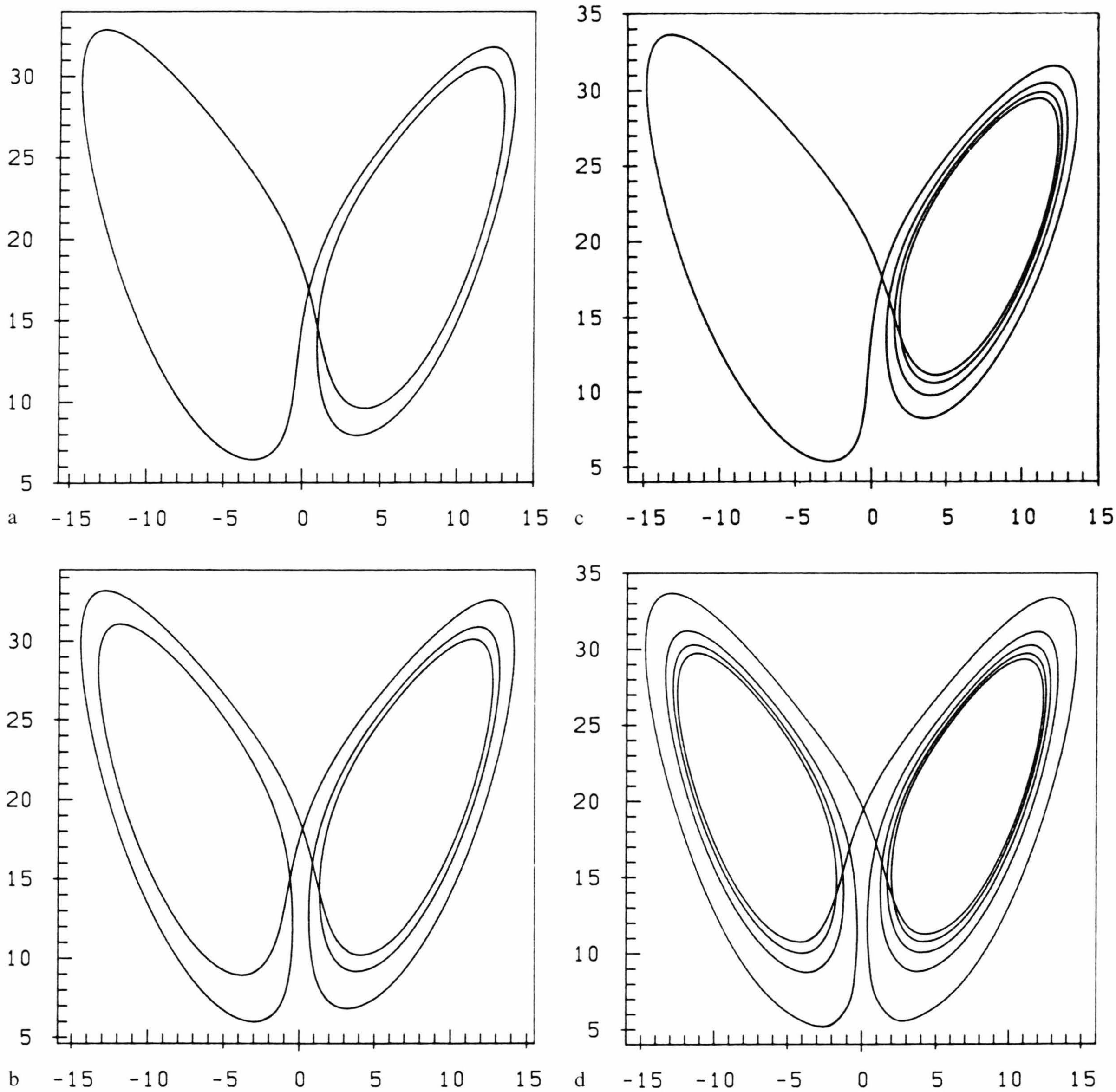

Fig. 1. Examples of unstable limit cycles of Lorenz-model. Parameters fixed at $\sigma = 10.0$, $b = 8/3$, $r = 22.0$ Only the projection on x (horizontal) z (vertical) plane is plotted: (a) $p^2 q$ orbit, (b) $p^3 q^2$ orbit, (c) $p^4 q$ orbit, (d) $p^5 q^4$ orbit.

additional harmonics set to 0 at the beginning. The higher the starting number N was chosen, the more complex limit cycles we got. The stability of the limit cycles was characterized by determining the Lyapunov exponents: one was always 0, one always negative, but the third one negative for stable and positive for unstable limit cycles, as expected [11]. We got a great number of stable and unstable limit cycles characterized in a nomenclature similar to Sparrow's, as $p^n q^m$, if there were n turns around the first (upper sign) and m turns around the second (lower sign) critical point ($x = \pm\sqrt{b(r-1)}$, $y = \pm\sqrt{b(r-1)}$, $z = r-1$). Limit cycles $p^n q^m$ exist in equivalent pairs $p^n q^m$ and $p^m q^n$, in addition there are symmetric cases with $n = m$. Some cases are shown in Fig. 1, the Lyapunov exponents are given in Table 1. All of the lowest cycles are unstable.

Table 1. Lyapunov exponents of the unstable limit cycles. Calculations were performed using double precision (complex) FORTRAN and using NAG routines on a CONVEX C 220.

Periodic orbit (limit cycle)	Number of harmonics	Real part of Lyapunov exponents (rounded to 5 digits after the period)		
p^2q	96	−12.83684	0.88252	0.00000
p^3q^2	96	−7.85271	0.85542	0.00000
p^4q	96	−8.04346	0.77128	0.00000
p^5q^4	96	−4.04143	0.76633	−0.00001

4. Discussion

It seems to us that unrestricted harmonic balance (UHB) is an approved method for determination of (unstable) limit cycles.

Explicitly the product formulation of truncated Fourier series, see paper I [3], provides a useful facility to master nonlinearities in the discussed model and so establishes the wide applicability of this method.

But there are other advantages: While all different methods, see e.g. [12], base on calculations in phase-space, harmonic balance works via Fourier transformation in frequency space. So the problem of finding limit cycles is transformed to the better understood problem of solving systems of nonlinear equations. A wide variety of highly optimized nonlinear least squares algorithms often developed for problems in engineering, and their program-codes in FORTRAN are now available for solving the equations, i.e. for determination of the Fourier coefficients. Further it should not be forgotten that Fourier coefficients provide a redundance-free representation of limit cycles. The Fourier coefficients of a periodic solution are a good basis for further calculations e.g. for differentiation in time, in our algorithm for a numerical refinement (more harmonics) or for calculation of Lyapunov coefficients. Remember that all other methods produce often a great number of points on the limit cycle in phase space. A combination of Runge-Kutta type methods and UHB via Fourier transformation of the periodic solution or time evolution of Fourier series, respectively, is possible.

Another advantage of this method is the effect that on scanning the parameter space the results for neighbouring sets of parameters are excellent starting points for the next set and need in many cases only a few iterations to adapt to the new slightly varied situation.

Acknowledgement

K. Neymeyr was partially supported by the "Studienstiftung des Deutschen Volkes".

[1] E. N. Lorenz, J. Atmos. Sci. **20,** 130 (1963).
[2] C. Sparrow, The Lorenz equations: Bifurcations, chaos and strange attractors, Applied mathematical sciences **41,** Springer Verlag, New York 1982.
[3] F. F. Seelig, Z. Naturforsch. **35a,** 1054 (1980).
[4] F. F. Seelig, J. Math. Biol. **12,** 187 (1981).
[5] F. F. Seelig and R. Füllemann, in: Numerical treatment of inverse problems in differential and integral equations (P. Deuflhard and E. Hairer, eds.), Birkheimer, Boston 1983.
[6] F. F. Seelig, Z. Naturforsch. **38a,** 636 (1983).
[7] F. F. Seelig, Z. Naturforsch. **38a,** 729 (1983).
[8] M. J. D. Powell, in: Numerical methods for nonlinear algebraic equations (P. Rabinowitz, ed.), Gordon & Breach, London 1970.
[9] J. E. Dennis, D. M. Gay, and R. E. Welsch, ACM T. Math. **7,** 348 (1981).
[10] J. E. Dennis, D. M. Gay, and R. E. Welsch, ACM T. Math. **7,** 369 (1981).
[11] E. A. Coddington, Theory of ordinary differential equations, McGraw Hill, New York 1955.
[12] T. S. Parker and L. O. Chua, Practical numerical algorithms for chaotic systems, Springer Verlag, New York 1989.